Waves in bubbly liquids with phase change

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Abstract

In a previous paper (C. Boutin, J.-L. Auriault, Acoustics of a bubbly fluid at large bubble concentration, Eur. J. Mech. B/Fluids, 12(3) (1993) 367–399), the homogenization technique was used to investigate how acoustic waves propagate in a bubbly fluid at finite concentration. Three different equivalent macroscopic behaviours were shown to exist, for "large"-, "medium"- and "small"-size bubble systems, respectively. In the present paper, we extend the analysis by taking into consideration possible phase change effects. We show that phase change effects are negligible in the case of large-size bubbles, whereas they strongly modify the medium-size bubble system behaviour. For small-size bubbles capillarity dominates the process.

1. Introduction

This paper deals with the mathematical modelling of wave propagation in bubbly fluids in presence of possible phase change. Interest in such investigations is accounted for by the fact that wave propagation characteristics drastically change when bubbles are generated in a liquid, because of the large change in compressibility. Appearance of bubbles can be caused by the wave perturbation itself when the liquid state stays in the vicinity of the bubble point. Obtaining mathematical modellings are of prime interest in many industries where two-phase systems are used, as well as in seismic detection of gas or oil reservoirs.

Wave propagation in bubbly fluids has been investigated for a long time. Steady or shock waves have been experimented in [3–8]. Phenomenological modellings can be found in [3] or in [9], where the relative displacement between the two phases is considered, and in [10–12] where thermal

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Nomenclature

c	wave celerity
$\mathscr{C}$	dimensionless number
C_{p}	heat capacity
d	diffusivity
$\mathbf{E}$	deformation tensor
$g(\mathbf{y}), h(\mathbf{y})$	temperature fields in the liquid and the vapour, respectively
$\mathbf{j}$	mass flux of vaporization–condensation
$\mathscr{J}$	Jacob number
k	thermal conductivity
K	rigidity
K_{c}	capillary membrane rigidity
$\tilde{K}$	bulk rigidity
l	characteristic length of the period
l^{visc}, l^{therm}	viscous and thermal layer thickness, respectively
L	macroscopic characteristic length
L^{φ}	vaporization enthalpy
$\mathscr{N}^{\mathrm{visc}}$, $\mathscr{N}^{\mathrm{therm}}$	dimensionless numbers for viscous and thermal effects, respectively
$\mathbf{n}$	unit normal to the bubble surface
p, P	pressure
p_{c}	capillary pressure
P^{c}	characteristic pressure
$\mathscr{P}r$	Prandtl number
r	bubble radius perturbation
R	bubble radius
T	temperature
T^{c}	characteristic temperature
$\mathbf{u}$	displacement
$\mathbf{U}$	macroscopic displacement
$\mathbf{U}^{\mathrm{c}}$	characteristic displacement
$\mathscr{W}$	Weber number
$\mathbf{x}$	dimensionless macroscopic space variable
$\mathbf{X}$	physical space variable
$\mathbf{y}$	dimensionless microscopic space variable

Greeks

β	bubble concentration
Δ	Laplacian operator
ε	small scale separation parameter
γ	specific heat ratio
Γ	bubble surface
λ	wavelength
μ	dynamic viscosity

ν	kinematic viscosity
ω	frequency
Ω	period
Ω_l, Ω_v	domains of the period occupied by the liquid, the vapour, respectively
ρ	density
ρ^c	characteristic density
$\tilde{\rho}$	bulk density
σ	surface tension
$\boldsymbol{\sigma}$	stress

effects are introduced. Many investigations are conducted by extrapolating the behaviour of a single bubble. A good review of such approaches can be found in [13]. Modellings of wave propagation were also investigated by scale change methods. Perfect fluid with finite or small gas concentration is studied in [14]. Non-linear behaviours including asymmetric bubble deformations were proposed in [15] and completed in [16] by adding viscous effects in a mixture at small concentration. Mixtures of two liquids were addressed in [17] and in [18] where capillarity is taken into consideration. Finally in [2], bubbly liquids at large concentration are considered in the presence of possible thermal exchanges.

The present paper is an extension of [2]; the phenomenon of phase change, which is of great importance in liquids near the bubble point is added. For this purpose we develop the analysis that is schematically introduced in [19]. The macroscopic equivalent models are determined by using the multiple scale expansion method [20]. In this method, we start from the bubble scale description. The method is based on the existence of two well-separated characteristic lengths, l and L, the ratio of which serves as a small parameter for the expansions. Macroscopic models are deduced from the local description, only, without any prerequisite concerning the macroscopic description. The volume averaging process is not arbitrarily introduced in the process; it is a consequence of the scale separation. Another advantage of the method is that it demonstrates, from a given local description, the existence or the non-existence of a macroscopic equivalent description. It also gives the domain of validity of the macroscopic description.

The bubbly liquid and the multiple scale expansion method are briefly presented in Section 2. After giving the physics at the bubble scale in Section 3, estimations of the different dimensionless numbers entering the local description are made in Section 4. That yields three characteristic different cases of interest corresponding to ''small'', ''medium'' and ''large'' bubble size, respectively, as in [2]. The corresponding dimensionless descriptions at the bubble scale are presented in Section 5. The macroscopic equivalent modellings are then investigated in Section 6 in the three cases.

2. Medium description and multiple scale homogenization process

We consider a liquid containing vapour bubbles at *finite* concentration. In order to make the homogenization method for periodic structures consistent with the investigation of the bubbly

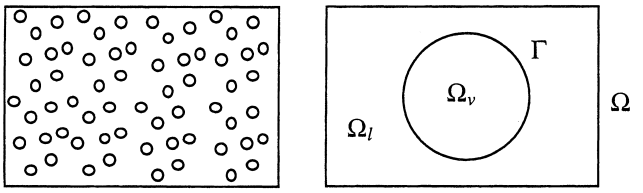

Fig. 1. (a) Bubbly fluid at finite concentration and (b) period Ω.

liquid, we assume the medium to be periodic. This assumption is actually not a restriction. At the bubble scale, consider the medium to be Ω-periodic and its characteristic length to be l. The liquid and the vapour occupy the domains Ω_l and Ω_v, respectively, and their common boundary is Γ (Fig. 1). We investigate how waves propagate when viscous, thermal, capillary and phase change effects are considered. The wave propagation introduces a second characteristic length L related to the wave celerity c, the frequency ω and the wavelength λ by, e.g.,

$$L = c/\omega = \lambda/2\pi.$$

We are looking for a macroscopic model that describes the wave propagation when $L \gg l$, i.e. when the two phase medium can be seen as a continuous medium. The scale separation $L \gg l$ is a necessary condition for an equivalent macroscopic description to be valid.

The existence of two well-separated characteristic lengths makes any one quantity depend a priori on two dimensionless space variables $\mathbf{x}$ and $\mathbf{y}$

$$\mathbf{x} = \frac{\mathbf{X}}{L}, \quad \mathbf{y} = \frac{\mathbf{X}}{l}, \quad \mathbf{x} = \varepsilon\mathbf{y}, \quad \frac{l}{L} = \varepsilon \ll 1, \tag{1}$$

where $\mathbf{X}$ is the physical space variable. The macroscopic description is obtained from the bubble scale description by using the method of multiple scale expansions. It consists (see [21]) firstly in rendering the local description dimensionless by using l as a characteristic length and evaluating the dimensionless numbers with respect to the powers of ε. Then any quantity Φ is looked for in the heuristic form

$$\Phi = \Phi(\mathbf{y}, \mathbf{x}) = \sum_n \varepsilon^n \Phi^n, \quad n \text{ integer}, \quad \mathbf{x} = \varepsilon\mathbf{y}. \tag{2}$$

Finally, by equating like power terms in equations, successive boundary value problems are obtained, that yield the macroscopic description (homogenizable situation). A dimensionless quantity Φ^* will be said to be of the order of ε^p if we have

$$\varepsilon^{p+1} \ll |\Phi^*| \ll \varepsilon^{p-1}.$$

The small parameter ε is not intrinsic to the particular investigated bubbly fluid, since from (1) it depends on the wavelength. It is clear that changing the wavelength changes ε, which then could

modify the value of the dimensionless numbers in function of the powers of ε, and therefore could result in a different macroscopic model. The liquid is assumed to be water and the gas in the bubbles is water vapour. The bubbly fluid is initially at rest at atmospheric pressure and the temperature is $T \approx 370$ K. We investigate a low frequency acoustic perturbation of this initial state.

3. Physics at the bubble scale

An acoustic perturbation of small amplitude and constant frequency is superimposed onto the initial equilibrium. The small parameter of the perturbation is assumed small enough to not interfere with the homogenization process. Any quantity Φ in the medium can be put in the form

$$\Phi_\alpha^{\mathrm{t}} = \Phi_\alpha^{\mathrm{e}} + \Phi_\alpha \exp(\mathrm{i}\omega t), \quad |\Phi_\alpha| \ll |\Phi_\alpha^{\mathrm{e}}|,$$

where the superscripts t and e stand for "total" and "equilibrium", respectively. $\alpha = \mathrm{l}$ is for the liquid and $\alpha = \mathrm{v}$ is for the vapour. Since the amplitude of the perturbation is small, the equations for the wave propagation are linearized.

3.1. Momentum balance

The two fluids are assumed viscous Newtonian. The momentum balance at constant frequency (linearized Navier–Stokes equation) is written as

$$-\mathbf{grad}\, p_\alpha + \mathrm{i}\omega\mu_\alpha(\mathbf{grad}\,\mathrm{div}\,\mathbf{u}_\alpha + \Delta\mathbf{u}_\alpha) = -\rho_\alpha^{\mathrm{e}}\omega^2\mathbf{u}_\alpha, \tag{3}$$

where p is the pressure perturbation, μ the viscosity, $\mathbf{u}$ the displacement and ρ is the density.

3.2. Heat balance

In the vapour and in the liquid the energy balances at constant frequency ω are respectively in the form

$$\mathrm{div}\,(k_{\mathrm{v}}\,\mathbf{grad}\,T_{\mathrm{v}}) = \mathrm{i}\omega(\rho_{\mathrm{v}}^{\mathrm{e}} C_{\mathrm{pv}} T_{\mathrm{v}} - p_{\mathrm{v}}), \tag{4}$$

$$\mathrm{div}\,(k_l\,\mathbf{grad}\,T_{\mathrm{l}}) = \mathrm{i}\omega\rho_{\mathrm{l}}^{\mathrm{e}} C_{\mathrm{pl}} T_{\mathrm{l}}. \tag{5}$$

where k is the thermal conductivity, T the temperature perturbation and $C_{\mathrm{p}\alpha}, \alpha = \mathrm{l}, \mathrm{v}$, is the specific heat at constant pressure.

3.3. Mass balance

The mass balance of the fluids is in the form

$$\operatorname{div}\left(\rho_\alpha^{\mathrm{t}}\frac{\partial \mathbf{u}}{\partial t}\right)=\frac{\partial \rho^{\mathrm{t}}}{\partial t},\quad \alpha=\mathrm{l},\mathrm{v}.$$

For the perturbation, the mass balance at constant frequency is linearized in the form

$$\rho_\alpha+\rho_\alpha^{\mathrm{e}}\operatorname{div}\mathbf{u}_\alpha=0,\quad \alpha=\mathrm{l},\mathrm{v}. \tag{6}$$

3.4. Equation of state of the vapour

The vapour is assumed as a perfect gas

$$p^{\mathrm{t}}\mathscr{V}^{\mathrm{t}}=n\mathscr{R}T^{\mathrm{t}}$$

that yields for the perturbation

$$p_{\mathrm{v}}=p_{\mathrm{v}}^{\mathrm{e}}\left(\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{v}}^{\mathrm{e}}}+\frac{T_{\mathrm{v}}}{T^{\mathrm{e}}}\right). \tag{7}$$

3.5. Conditions on the bubble surface Γ

Because of capillarity, the stress perturbation $\boldsymbol{\sigma}$ is discontinuous on Γ

$$(\boldsymbol{\sigma}_{\mathrm{v}}-\boldsymbol{\sigma}_{\mathrm{l}})\cdot\mathbf{n}=p_{\mathrm{c}}\mathbf{n},\quad p_{\mathrm{c}}=\frac{2\sigma r}{R^2}. \tag{8}$$

Vector $\mathbf{n}$ is a unit normal to Γ, p_{c} the capillary pressure perturbation, σ the surface tension, R the bubble radius at rest and r is the perturbation of the bubble curvature.

As in [2], the surface Γ is assumed to be at thermodynamic equilibrium

$$T_{\mathrm{v}}=T_{\mathrm{l}}=T_\Gamma. \tag{9}$$

However, due to phase change the temperature T_Γ^{t} is now given by the Clausius–Clapeyron relation

$$\frac{\mathrm{d}T_\Gamma^{\mathrm{t}}}{\mathrm{d}p_{\mathrm{v}}^{\mathrm{t}}}=\frac{T_\Gamma^{\mathrm{t}}}{L^\varphi\rho_{\mathrm{v}}^{\mathrm{t}}}.$$

For the perturbation, we get

$$T_\Gamma=p_{\mathrm{v}}\frac{T^{\mathrm{e}}}{L^\varphi\rho_{\mathrm{v}}^{\mathrm{e}}}, \tag{10}$$

where L^φ is the enthalpy of vaporization.

The heat flux perturbation is discontinuous on Γ because of the phase change

$$k_l \, \mathbf{grad}\, T_l \cdot \mathbf{n} = k_v \, \mathbf{grad}\, T_v \cdot \mathbf{n} - L^{\varphi} \mathbf{j} \cdot \mathbf{n}. \tag{11}$$

where $\mathbf{j}$ is the mass flux of vaporization–condensation at the vapour–liquid boundary Γ.

The last condition on Γ concerns displacements $\mathbf{u}$ which is also discontinuous because of the phase change

$$i\omega \mathbf{u}_\Gamma \cdot \mathbf{n} = i\omega \mathbf{u}_l \cdot \mathbf{n} - \frac{\mathbf{j} \cdot \mathbf{n}}{\rho_l^e} = i\omega \mathbf{u}_v \cdot \mathbf{n} - \frac{\mathbf{j} \cdot \mathbf{n}}{\rho_v^e}. \tag{12}$$

4. Estimations

With a view to make dimensionless the bubble scale description, we investigate the orders of magnitude of the different terms in Eqs. (3)–(12). To illustrate the analysis, we consider a particular bubbly fluid composed of water and water vapour bubbles. Changing the liquid–gas mixture or changing the equilibrium state before perturbation could change the orders of magnitude of the different terms in Eqs. (3)–(12). However, the macroscopic behaviours that are described in the following would be preserved for corresponding appropriate bubble sizes and frequencies. The concentration β of the bubble is finite, $\beta = O(1)$. The medium is initially at rest, and the liquid is at the temperature $T^e \approx 370$ K, and at atmospheric pressure, $P^e \approx 10^5$ Pa. The characteristic length l is considered as of the same order of magnitude as the bubble diameter, $l = O(2R)$. We investigate an acoustic perturbation characterized by a small scale separation parameter $\varepsilon \ll 1$ and we consider low acoustical frequencies, $1\ s^{-1} < \omega < 500\ s^{-1}$.

4.1. Material constants

We use the typical values shown in Table 1 for the liquid and the vapour. The enthalpy of vaporization is $L^{\varphi} = 2.5 \times 10^5$ J/kg. The surface tension is $\sigma = 0.075$ N/m. Notice that the Jacob number $\mathscr{J}$ and the vapour Prandtl number $\mathscr{P}r_v$ are of order 1

$$\mathscr{J} = \frac{C_p T^e}{\mathscr{L}} = O(1), \quad \mathscr{P}r_v = \frac{\mu_v C_{pv}}{k_v} = O(1).$$

Table 1
Medium characteristic constants

Medium	Liquid	Vapour
Density ρ^e (Kg/m^3)	10^3	1.2
Rigidity K (Pa)	2×10^9	1.4×10^5
Dynamic viscosity μ (Pa s)	10^{-3}	20×10^{-6}
Kinematic viscosity v (m^2/s)	10^{-6}	15×10^{-6}
Conductivity k (W/K m)	0.6	0.026
Heat capacity C_p (J/K kg)	4.18×10^3	10^3
Diffusivity d (m^2/s)	1.4×10^{-7}	2.1×10^{-5}

4.2. The three characteristic bubble sizes

Viscous and thermal effects are characterized by the viscous and thermal layer thickness l^{visc} and l^{therm} in the liquid and in the vapour, respectively

$$l_\alpha^{\mathrm{visc}} = \sqrt{\frac{\nu_\alpha}{\omega}}, \quad l_\alpha^{\mathrm{therm}} = \sqrt{\frac{d_\alpha}{\omega}}, \quad \alpha = \mathrm{l}, \mathrm{v}.$$

These thicknesses are to be compared to the bubble radius R. It is easy to check that they are all of similar order of magnitude relatively to $\varepsilon \ll 1$

$$l_\alpha^{\mathrm{visc}} = \mathrm{O}(l_\alpha^{\mathrm{therm}}) \quad \alpha = \mathrm{l}, \mathrm{v}.$$

On the other hand, capillary effects are characterized by the capillary membrane rigidity K_{c}

$$K_{\mathrm{c}} = \frac{2\sigma}{3R}$$

to be compared to the rigidity P^{e}/β of the vapour–liquid mixture in absence of thermal and capillary effects. Therefore, by following [2], three cases of interest are pointed out:

- *Large-size bubbles*. The layer thickness is much smaller than the bubble radius. Outside the boundary layer, the heat transfer is negligible and the process is *adiabatic*. The rigidity that is due to capillarity is very small

$$l_\alpha^{\mathrm{visc}} = \mathrm{O}(l_\alpha^{\mathrm{therm}}) \ll R, \quad T_\alpha = \mathrm{O}\left(\frac{p_\alpha}{\rho_\alpha^{\mathrm{e}} C_{\mathrm{p}\alpha}}\right), \quad K_{\mathrm{c}} \ll \frac{P^{\mathrm{e}}}{\beta}. \tag{13}$$

  As the pressure perturbations are of the same order in the liquid and the vapour, when using the heat capacity values shown in Table 1, it is found that

$$T_{\mathrm{v}} = \mathrm{O}(\varepsilon^{-1} T_{\mathrm{l}}).$$

- *Medium-size bubbles*. The layer thickness is of the order of magnitude as the bubble radius. Heat transfer occurs in the entire bubble volume. The heat transfer is *transient*. However, the above temperature estimation remains valid, $T_{\mathrm{v}} = \mathrm{O}(\varepsilon^{-1} T_{\mathrm{l}})$. Capillarity still remains small

$$l_\alpha^{\mathrm{visc}} = \mathrm{O}(l_\alpha^{\mathrm{therm}}) = \mathrm{O}(R), \quad K_{\mathrm{c}} \ll \frac{P^{\mathrm{e}}}{\beta}. \tag{14}$$

- *Small-size bubbles*. The layer thickness is much larger than the bubble radius. The process is *isothermal*, $T_{\mathrm{v}} = \mathrm{O}(T_{\mathrm{l}})$. Capillarity becomes important

$$l_\alpha^{\mathrm{visc}} = \mathrm{O}(l_\alpha^{\mathrm{therm}}) \gg R, \quad K_{\mathrm{c}} = \mathrm{O}\left(\frac{P^{\mathrm{e}}}{\beta}\right). \tag{15}$$

4.3. Phase change effects, thermal flux condition on Γ

Let us investigate the heat flux condition (11) on Γ. An estimate of the heat flux ratio is in the form

$$\left(\frac{|k_l \,\mathbf{grad}\, T_\mathrm{l}|}{|k_\mathrm{v} \,\mathbf{grad}\, T_\mathrm{v}|}\right)_\Gamma = \mathrm{O}\left(\frac{k_\mathrm{l} l_\mathrm{v}^{\mathrm{therm}}}{k_\mathrm{v} l_\mathrm{l}^{\mathrm{therm}}} \frac{\delta T_\mathrm{l}}{\delta T_\mathrm{v}}\right),$$

where $\delta T = |T_\alpha - T_\Gamma|$ stands for the temperature increment in the boundary layers. For large bubbles the process in the vapour is *adiabatic*. We get from (10) and (13) and from the adopted characteristic values that

$$\frac{T_\Gamma}{T_\mathrm{v}} = \frac{T^\mathrm{e} C_\mathrm{pv}}{L^\varphi} = \mathscr{J}_\mathrm{v} = \mathrm{O}(1).$$

This estimation remains valid for medium- and small-size bubbles. As shown in Section 4.2, we have for large- and medium-size bubbles

$$T_\mathrm{l} = \mathrm{O}(\varepsilon T_\mathrm{v}) = \mathrm{O}(\varepsilon T_\Gamma),$$

and for small-size bubbles

$$T_\mathrm{l} = \mathrm{O}(T_\mathrm{v}) = \mathrm{O}(T_\Gamma).$$

Whatever the bubble size is, we get from the above estimations that

$$\frac{\delta T_\mathrm{l}}{\delta T_\mathrm{v}} = \mathrm{O}(1), \quad \frac{k_\mathrm{l} l_\mathrm{v}^{\mathrm{therm}}}{k_\mathrm{v} l_\mathrm{l}^{\mathrm{therm}}} = \sqrt{\frac{k_\mathrm{l}\rho_\mathrm{l} C_\mathrm{pl}}{k_\mathrm{v}\rho_\mathrm{v} C_\mathrm{pv}}} = \mathrm{O}(\varepsilon^{-1}).$$

That yields

$$\left(\frac{|k_\mathrm{l} \,\mathbf{grad}\, T_\mathrm{l}|}{|k_\mathrm{v} \,\mathbf{grad}\, T_\mathrm{v}|}\right)_\Gamma = \mathrm{O}(\varepsilon^{-1}).$$

Therefore the mass flux $\mathbf{j}\cdot\mathbf{n}$ is mainly generated by the heat flux in the liquid. This result is in agreement with the estimation in [13]

$$L^\varphi \mathbf{j}\cdot\mathbf{n} = \mathrm{O}(-k_\mathrm{l} \,\mathbf{grad}_y\, T_\mathrm{l}\cdot\mathbf{n}). \tag{16}$$

4.4. Temperature fields in the fluid

Let us investigate the temperature field in the liquid, when $l_l^{\mathrm{therm}} = \mathrm{O}(R)$. The boundary value problem for T_l is in the form

$$\operatorname{div}(k_{\mathrm{l}}\,\mathbf{grad}\,T_{\mathrm{l}}) = \mathrm{i}\omega\rho_{\mathrm{l}}^{\mathrm{e}}C_{\mathrm{pl}}T_{\mathrm{l}} \quad \text{in } \Omega_{\mathrm{l}}, \tag{17}$$

$$T_{\mathrm{l}} = T_{\Gamma} = p_{\mathrm{v}}\frac{T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}} \quad \text{on } \Gamma, \tag{18}$$

where T_{l} is Ω-periodic. It will be seen that p_{v} does not depend on the local variable in the period, at the first order of approximation. Therefore, from relation (18) T_{Γ} is also a constant on the period. After introducing the new unknown w

$$w = T_{\mathrm{l}} - T_{\Gamma},$$

we get

$$\operatorname{div}(k_{\mathrm{l}}\,\mathbf{grad}\,w) = \mathrm{i}\omega\rho_{\mathrm{l}}^{\mathrm{e}}C_{\mathrm{pl}}(w + T_{\Gamma}) \quad \text{in } \Omega_{\mathrm{l}},$$

$$w = 0 \quad \text{on } \Gamma.$$

This is a problem already solved in [22]. The temperature is in the form

$$w = -h(\mathbf{y})\ T^{\mathrm{e}}p_{\mathrm{v}}\frac{1}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}},$$

$$T_{\mathrm{l}} = \frac{p_{\mathrm{v}}T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}}(1 - h(\mathbf{y})), \tag{19}$$

where $h(\mathbf{y})$ is a complex function of the dimensionless frequency $\omega/\omega_{l}^{\mathrm{therm}}$, with $\omega_{\mathrm{l}}^{\mathrm{therm}} = d_{\mathrm{l}}/R^2$. As shown in [22],

$$\lim_{\omega\to 0} h(\mathbf{y}) = 0, \quad \lim_{\omega\to\infty} h(\mathbf{y}) = 1.$$

For small-size bubbles, h is small. For medium-size bubbles, $h = \mathrm{O}(1)$. For large-size bubbles, $1 - h \ll 1$ and the temperature perturbation in the liquid is obtained as $T_{\mathrm{l}} = 0$. In this later case, a boundary layer is to be introduced to match this temperature field in the liquid to the boundary condition (18).

4.5. Temperature field in the vapour

Let us first consider the boundary value problem for T_{v} in the case of medium-size bubbles

$$\operatorname{div}(k_{\mathrm{v}}\,\mathbf{grad}\,T_{\mathrm{v}}) = \mathrm{i}\omega(\rho_{\mathrm{v}}^{\mathrm{e}}C_{\mathrm{pv}}\ T_{\mathrm{v}} - p_{\mathrm{v}}) \quad \text{in } \Omega_{\mathrm{v}}, \tag{20}$$

$$T_{\mathrm{v}} = T_{\Gamma} = p_{\mathrm{v}}\frac{T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}} \quad \text{on } \Gamma. \tag{21}$$

We proceed as for the liquid temperature by introducing the new unknown w

$$w = T_{\mathrm{v}} - T_{\Gamma}.$$

We obtain

$$\mathrm{div}(k_{\mathrm{v}}\,\mathbf{grad}\,w) = \mathrm{i}\omega(\rho_{\mathrm{v}}^{\mathrm{e}} C_{\mathrm{pv}}(w + T_{\Gamma}) - p_{\mathrm{v}}) \quad \text{in } \Omega_{\mathrm{v}},$$

$$w = 0 \quad \text{on } \Gamma.$$

This problem resembles the problem for the liquid temperature. It is already solved in [22]. With the notations in [2], the temperature is in the form

$$w = -g(\mathbf{y})\left(T_{\Gamma} - \frac{p_{\mathrm{v}}}{\rho_{\mathrm{v}}^{\mathrm{e}} C_{\mathrm{pv}}}\right) = g(\mathbf{y}) T^{\mathrm{e}} p_{\mathrm{v}}\left[\left(1 - \frac{1}{\gamma}\right)\frac{1}{p_{\mathrm{v}}^{\mathrm{e}}} - \frac{1}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}}\right],$$

$$T_{\mathrm{v}} = g(\mathbf{y}) T^{\mathrm{e}} p_{\mathrm{v}}\left(1 - \frac{1}{\gamma}\right)\frac{1}{p_{\mathrm{v}}^{\mathrm{e}}} + \frac{p_{\mathrm{v}} T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}}(1 - g(\mathbf{y})). \tag{22}$$

When $L^{\varphi} \to \infty$, we recover the temperature field in the absence of phase change. $g(\mathbf{y})$ is a complex function of the dimensionless frequency $\omega/\omega_{\mathrm{v}}^{\mathrm{therm}}$, with $\omega_{\mathrm{v}}^{\mathrm{therm}} = d_{\mathrm{v}}/R^2$. As shown in [22],

$$\lim_{\omega \to 0} g(\mathbf{y}) = 0, \quad \lim_{\omega \to \infty} g(\mathbf{y}) = 1.$$

For small-size bubbles, the boundary value problem becomes

$$\mathrm{div}(k_{\mathrm{v}}\,\mathbf{grad}\,T_{\mathrm{v}}) = 0 \quad \text{in } \Omega_{\mathrm{v}}, \tag{23}$$

$$T_{\mathrm{v}} = T_{\Gamma} = p_{\mathrm{v}}\frac{T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}} \quad \text{on } \Gamma. \tag{24}$$

That yields

$$T_{\mathrm{v}} = \frac{p_{\mathrm{v}}\ T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}}. \tag{25}$$

Finally, for large-size bubbles, T_{v} verifies

$$\rho_{v}^{\mathrm{e}} C_{\mathrm{pv}} T_{\mathrm{v}} - p_{\mathrm{v}} = 0 \quad \text{in } \Omega_{\mathrm{v}}, \tag{26}$$

$$T_{\mathrm{v}} = T_{\Gamma} = p_{\mathrm{v}}\frac{T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}} \quad \text{on } \Gamma. \tag{27}$$

Relation (22) becomes

$$T_{\mathrm{v}} = \frac{p_{\mathrm{v}}}{\rho_{\mathrm{v}}^{\mathrm{e}} C_{\mathrm{pv}}} = T^{\mathrm{e}} p_{\mathrm{v}} \left(1 - \frac{1}{\gamma}\right) \frac{1}{p_{\mathrm{v}}^{\mathrm{e}}}. \tag{28}$$

To match (28) with (27), a boundary layer should be introduced. Eqs. (25) and (28) are obtained from (22) by letting $g(\mathbf{y}) \sim 0$ and $g(\mathbf{y}) \sim 1$, respectively. Phase change has no effect for large bubbles.

4.6. Bulk rigidity of the bubbly fluid

First, let us evaluate the vaporization–condensation flux. Denoting by $\mathbf{n}_{\mathrm{l}}$ the outward normal to Ω_{l}, we obtain from (16)

$$\begin{aligned} L^{\varphi} \int_{\Gamma} \mathbf{j} \cdot \mathbf{n}_{\mathrm{l}} \, \mathrm{d}\Gamma &= -\int_{\Gamma} k_{\mathrm{l}} \mathbf{grad}_y T_{\mathrm{l}} \cdot \mathbf{n}_{\mathrm{l}} \, \mathrm{d}\Gamma = -\int_{\Omega_{\mathrm{l}}} \mathrm{div}(k_{\mathrm{l}} \mathbf{grad}_y T_{\mathrm{l}}) \, \mathrm{d}\Omega = -\mathrm{i}\omega \rho_{\mathrm{l}}^{\mathrm{e}} C_{\mathrm{pl}} \int_{\Omega_{\mathrm{l}}} T_{\mathrm{l}} \, \mathrm{d}\Omega \\ &= \mathrm{i}\omega \Omega_{\mathrm{l}} \frac{p_{\mathrm{v}} \rho_{\mathrm{l}}^{\mathrm{e}} C_{\mathrm{pl}} T^{\mathrm{e}}}{L^{\varphi} \rho_{\mathrm{v}}^{\mathrm{e}}} (1 - H), \end{aligned} \tag{29}$$

$$H = \frac{1}{\Omega_l} \int_{\Omega_l} h \, \mathrm{d}\Omega,$$

where we used the divergence theorem and (19). For small-size bubbles we have $H \ll 1$. For medium-size bubbles, the thermal layer thickness is $l_{\mathrm{l}}^{\mathrm{therm}} = \mathrm{O}(R)$ and therefore $1 - H = \mathrm{O}(1)$. For large-size bubbles, we obtain $l_{\mathrm{l}}^{\mathrm{therm}} \ll R$ and therefore $1 - H \ll 1$. The vaporization–condensation flux $\mathbf{j} \cdot \mathbf{n}$ becomes much smaller as bubble radius increases. From the above estimation we get

$$\int_{\Gamma} \mathbf{j} \cdot \mathbf{n} \, \mathrm{d}\Gamma = \mathrm{O}\left(\frac{\omega p_{\mathrm{v}} \Omega_{\mathrm{l}} \rho_{\mathrm{l}}^{\mathrm{e}} C_{\mathrm{pl}} T^{\mathrm{e}}}{L^{\varphi^2} \rho_{\mathrm{v}}^{\mathrm{e}}} (1 - H) \right) = \mathrm{O}\left(\mathscr{C} \omega \frac{p_{\mathrm{v}}^0}{P^{\mathrm{e}}} \Omega_l \rho_{\mathrm{v}}^{\mathrm{e}} (1 - H) \right), \tag{30}$$

where

$$\mathscr{C} = \frac{\rho_l^{\mathrm{e}} C_{\mathrm{pl}} T^{\mathrm{e}} P^{\mathrm{e}}}{\left(\rho_{\mathrm{v}}^{\mathrm{e}} L^{\varphi}\right)^2} \approx 620 \gg 1.$$

The corresponding amplitude θ_{φ} of the relative vapour volume produced by vaporization–condensation is in the form

$$\theta_{\varphi} = -\mathbf{j} \cdot \mathbf{n} \frac{4\pi R^2}{\omega \rho_{\mathrm{v}}^{\mathrm{e}} \Omega_{\mathrm{v}}} = \mathrm{O}\left(-\frac{\Omega_{\mathrm{l}}}{\Omega_{\mathrm{v}}} \mathscr{C} \frac{p_{\mathrm{v}}}{P^{\mathrm{e}}} (1 - H) \right).$$

The total relative vapour volume change θ_v is the sum of the relative volume change θ_{pT} in the absence of phase change plus the relative volume change θ_φ due to phase change. θ_{pT} can be estimated from (7)

$$\theta_{pT} = O\left(-\frac{p_v}{p_v^e} + \frac{T_v}{T^e} \right) = O\left(-\frac{p_v}{p_v^e} + \frac{T_\Gamma}{T^e} \right).$$

Then we get

$$\theta_v = O(\theta_{pT} + \theta_\varphi) = O\left(-\frac{p_v}{P^e} \left(\frac{P^e}{p_v^e} - \frac{P^e}{\mathscr{L}^\varphi \rho_v^e} + \frac{1-\beta}{\beta} \mathscr{C}(1-H) \right) \right). \tag{31}$$

Now, the stress discontinuity (8) on Γ yields

$$p_l = p_v + \frac{2\sigma r}{R^2}, \tag{32}$$

where p_v is related to θ_v by Eq. (31) and r is given by

$$r = \frac{R\theta_v}{3}. \tag{33}$$

When eliminating p_v^0 and r between 31, 32 and 33 and denoting by $\theta = \beta\theta_v$, the bulk deformation of the period, we obtain

$$p_l = -\tilde{K}\theta$$

with the bulk rigidity defined by

$$\tilde{K} = O\left(\frac{P^e}{\beta\left(\frac{P^e}{P^e + 2\sigma/R} - \frac{P^e}{\mathscr{L}^\varphi \rho_v^e} + \frac{1-\beta}{\beta}\mathscr{C}(1-H)\right)} - \frac{2\sigma}{3\beta R} \right). \tag{34}$$

Depending on the values of H and $K_c = 2\sigma/3R$, three behaviours are again distinguished:

- *Large-size bubbles*. K_c and the phase change contribution are negligible

$$\tilde{K} = O\left(\frac{P^e}{\beta} \right) = O(P^e).$$

  We recover the order of magnitude of the rigidity γP^e of an adiabatic mixture.
- *Medium-size bubbles*. Phase change effects are dominant

$$\tilde{K} = O\left(\frac{P^e}{(1-\beta)\mathscr{C}(1-H)} \right) \ll P^e.$$

- *Small-size bubbles*. Capillary effects become preponderant.

$$\tilde{K} = \mathrm{O}\left(-\frac{2\sigma}{3\beta R} \right).$$

The bulk rigidity is negative. The perturbation does not propagate.

The real and imaginary parts K_r and K_i of the rigidity K are shown in Figs. 2 and 3, respectively, for $\omega = 100$ Hz and $\beta = 0.2$. Note that the large-size bubble behaviour is not realistic in the present investigated case. Realistic large-size behaviours are obtained for higher frequencies. The thermal memory function H was approximated in the following form [1]:

$$H = \frac{\mathrm{i}\omega^*}{\mathrm{i}\omega^* + \sqrt{F_t^2 + \mathrm{i}\omega^*}},$$

where $\omega^* = \omega/\omega_\mathrm{t}$ is a dimensionless frequency, where the characteristic frequency ω^* is given by

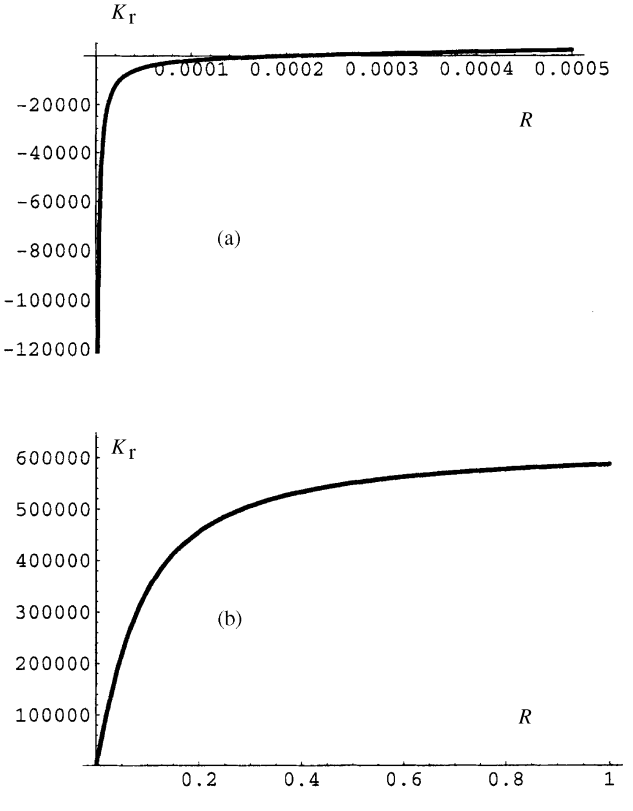

Fig. 2. Real part K_r (Pa) of the bulk rigidity versus the bubble radius R (m), $\omega = 100$ Hz: (a) small-size bubble and (b) medium-size and large-size bubble.

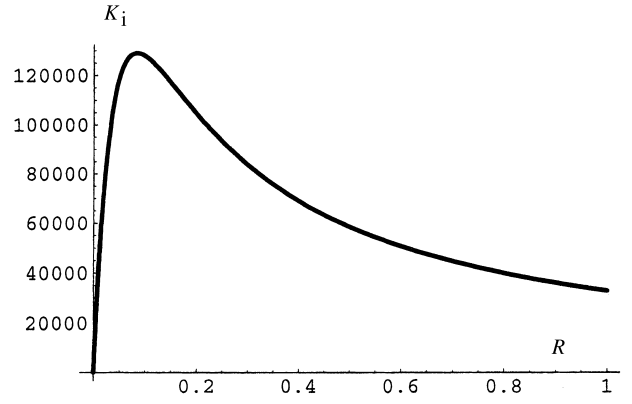

Fig. 3. Imaginary part K_i (Pa) of the bulk rigidity as a function of R (m), $\omega = 100$ Hz.

$$\omega_t = \frac{d_l}{\Lambda_t^2}, \quad \Lambda_t = \frac{\omega_l}{\Gamma} = \frac{1-\beta}{3\beta}R.$$

where Λ_t is a geometric characteristic length. F_t is a shape factor. $F_t = 2$ for cylindrical pores and $F_t = 5/3$ for spherical pores. We used the value $F_t = 2$ to calculate.

4.7. Scale separation

The characteristic size of the period is related to the bubble radius by

$$l = R\left(\frac{4\pi}{3\beta}\right)^{1/3},$$

whereas the macroscopic characteristic length is given by

$$L = \frac{\lambda}{2\pi} = \frac{|c|}{\omega} = \frac{1}{\omega}\left(\frac{\tilde{K}}{\tilde{\rho}}\right)^{1/2},$$

where c is the wave celerity, that is of pure imaginary value in case of small-size bubbles, and $\tilde{\rho} = (1-\beta)\rho_l^e$ is the bulk density of the mixture. Considering a bubble concentration $\beta \approx 0.1$ and $\omega \approx 50\ s^{-1}$ yields for medium-size bubbles

$$\varepsilon = \frac{l}{L} \approx 10^{-2}.$$

This value of the small scale separation parameter is preserved for large-size bubbles on the condition that, according to Eq. (13), we have $R = O(\varepsilon^{-1/2}\sqrt{v/\omega})$, $\omega \approx 500$ and for small-size bubbles on the condition that, according to Eq. (15), we have $R = O(\varepsilon^{1/2}\sqrt{v/\omega})$, $\omega \approx 5$. For simplicity, we adopt these values in the following.

4.8. Displacement and radius perturbation

The perturbation is assumed to be small. Therefore the bulk volume perturbation θ itself is small. We consider

$$\theta = \mathrm{O}\left(\frac{u}{L}\right) = \mathrm{O}(\varepsilon^2),$$

where u is for the displacement. Therefore, since $r = \mathrm{O}(R\theta/3)$, we get

$$r = \mathrm{O}(\varepsilon u) = \mathrm{O}(\varepsilon^2 R) = \mathrm{O}(\varepsilon^3 L).$$

5. Dimensionless bubble-scale description

We use l as the characteristic length to make dimensionless the equations

$$\mathbf{X} = l\mathbf{y}.$$

With the above estimations, we introduce the following characteristic values for the displacement, density, pressure and temperature, respectively

$$U^{\mathrm{c}} = \varepsilon R, \quad \rho^{\mathrm{c}} = \tilde{\rho}, \quad P^{\mathrm{c}} = \mathrm{O}\left(\tilde{K}\frac{U^{\mathrm{c}}}{L}\right), \quad T^{\mathrm{c}} = P^{\mathrm{c}}\frac{T^{\mathrm{e}}}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}} = \mathrm{O}(T_\Gamma).$$

With the help of Table 1 and the above characteristic values we define dimensionless quantities shown by a star

$$\mathbf{u}_l = U^{\mathrm{c}}\mathbf{u}_{\mathrm{l}}^*, \mathbf{u}_{\mathrm{v}} = U^{\mathrm{c}}\mathbf{u}_{\mathrm{v}}^*, \quad \rho_{\mathrm{l}}^{\mathrm{e}} = \rho^{\mathrm{c}}\rho_{\mathrm{l}}^{\mathrm{e}*}, \quad \rho_{\mathrm{v}}^{\mathrm{e}} = \varepsilon\rho^{\mathrm{c}}\rho_{\mathrm{v}}^{\mathrm{e}*},$$

$$p_{\mathrm{l}} = P^{\mathrm{c}}p_{\mathrm{l}}^*, \quad p_{\mathrm{v}} = P^{\mathrm{c}}p_{\mathrm{v}}^*, \quad T = T^{\mathrm{c}}T^*.$$

To avoid cumbersome notation we shall drop the stars that show dimensionless quantities.

5.1. Momentum balance

Viscous effects introduce the following dimensionless number

$$\mathscr{N}^{\mathrm{visc}} = \mathrm{O}\left(\left(\frac{l^{\mathrm{visc}}}{R}\right)^2\right),$$

where $l_{\mathrm{l}}^{\mathrm{visc}}$ and $l_{\mathrm{v}}^{\mathrm{visc}}$ are the viscous layer thickness in the liquid and the vapour, respectively. They are of the same order of magnitude

$$\mathscr{N}_{\mathrm{l}}^{\mathrm{visc}} = \mathrm{O}(\mathscr{N}_{\mathrm{v}}^{\mathrm{visc}}) = \mathscr{N}^{\mathrm{visc}}.$$

where $\mathscr{N}^{\mathrm{visc}}$ is the inverse of the acoustic Reynolds number. We obtain $\mathscr{N}^{\mathrm{visc}} = \mathrm{O}(\varepsilon)$, $\mathrm{O}(1)$ and $\mathrm{O}(\varepsilon^{-1})$ for large, medium and small-size bubbles, respectively. The dimensionless momentum balances are in the form

$$\varepsilon^{-1}\,\mathbf{grad}\,p_{\mathrm{l}} - \mathrm{i}\omega\mu_{\mathrm{l}}\mathscr{N}^{\mathrm{visc}}(\mathbf{grad}\ \mathrm{div}\,\mathbf{u}_{\mathrm{l}} + \Delta\mathbf{u}_{\mathrm{l}}) = \rho_{\mathrm{l}}^{\mathrm{e}}\omega^2\mathbf{u}_{\mathrm{l}}, \tag{35}$$

$$\varepsilon^{-1}\,\mathbf{grad}\,p_{\mathrm{v}} - \mathrm{i}\varepsilon\omega\mu_{\mathrm{v}}\mathscr{N}^{\mathrm{visc}}\,(\mathbf{grad}\ \mathrm{div}\,\mathbf{u}_{\mathrm{v}} + \Delta\mathbf{u}_{\mathrm{v}}) = \varepsilon\rho_{\mathrm{v}}^{\mathrm{e}}\omega^2\mathbf{u}_{\mathrm{v}}. \tag{36}$$

Corresponding to Eqs. (35) and (36), the stress perturbations take the forms

$$\boldsymbol{\sigma}_{\mathrm{l}} = -p_{\mathrm{v}}\mathbf{I} + 2\varepsilon\mathrm{i}\omega\mathscr{N}^{\mathrm{visc}}\mathbf{E}(\mathbf{u}_{\mathrm{l}}),$$
$$\boldsymbol{\sigma}_{\mathrm{l}} = -p_{\mathrm{v}}\mathbf{I} + 2\varepsilon^2\mathrm{i}\omega\mathscr{N}^{\mathrm{visc}}\mathbf{E}(\mathbf{u}_{\mathrm{v}}),$$

where $\mathbf{E}$ is the linear strain

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right).$$

5.2. Heat balance

The energy balances introduce the dimensionless numbers $\mathscr{N}_{\alpha}^{\mathrm{therm}}$

$$\mathscr{N}_{\alpha}^{\mathrm{therm}} = \frac{k_\alpha}{\rho_\alpha^{\mathrm{e}} C_{\mathrm{p}\alpha} R^2\omega} = \frac{d_\alpha}{\omega R^2} = \left(\frac{l_\alpha^{\mathrm{therm}}}{R}\right)^2, \quad \alpha = \mathrm{l}, \mathrm{v}.$$

From the values in Table 1 we have

$$\mathscr{N}_{l}^{\mathrm{therm}} = O(\varepsilon\mathscr{N}_{\mathrm{v}}^{\mathrm{therm}}). \tag{37}$$

Therefore the temperature field is quasi-static in the liquid. However, for larger generality we consider a transient regime in the liquid and we use equal estimation of the $\mathscr{N}_{\alpha}^{\mathrm{therm}}$'s

$$\mathscr{N}_{l}^{\mathrm{therm}} = \mathrm{O}(\mathscr{N}_{\mathrm{v}}^{\mathrm{therm}}) = \mathrm{O}(\mathscr{N}^{\mathrm{visc}}). \tag{38}$$

Then we obtain $\mathscr{N}^{\mathrm{therm}} = \mathrm{O}(\varepsilon)$, $\mathrm{O}(1)$ and $\mathrm{O}(\varepsilon^{-1})$ for large-, medium- and small-size bubbles, respectively. In the vapour the corresponding dimensionless energy balance takes the form

$$\mathscr{N}_{\mathrm{v}}^{\mathrm{therm}}\mathrm{div}\,(k_{\mathrm{v}}\,\mathbf{grad}\,T_{\mathrm{v}}) = \mathrm{i}\omega(\rho_{\mathrm{v}}^{\mathrm{t}} C_{\mathrm{pv}}\,T_{\mathrm{v}} - p_{\mathrm{v}}) \tag{39}$$

and in the liquid we get

$$\mathscr{N}_l^{\text{therm}} \operatorname{div}(k_\text{l} \mathbf{grad}\, T_\text{l}) = \mathrm{i}\omega \rho_\text{l}^\text{t} C_\text{pl} T_\text{l}. \tag{40}$$

5.3. Stress discontinuity on Γ

Capillarity is measured by the Weber number $\mathscr{W}$

$$\mathscr{W}^{-1} = \frac{p_\text{c}}{P^\text{c}} = \frac{2\sigma r}{R^2} \frac{L}{\tilde{K} U^\text{c}} = \frac{K_\text{c}}{\tilde{K}} \frac{3rL}{RU^\text{c}}, \quad K_\text{c} = \frac{2\sigma}{3R}.$$

K_c is for the capillary membrane rigidity. $K_\text{c}/\tilde{K}$ is a rigidity dimensionless number. $3rL/RU^c$ is a dimensionless number related to the bubble deformation. From the above estimations it is O(1). Therefore the dimensionless capillary condition on Γ is in the form

$$(\boldsymbol{\sigma}_\text{v} - \boldsymbol{\sigma}_\text{l}) \cdot \mathbf{n} = \mathscr{W}^{-1} p_\text{c} \mathbf{n}, \tag{41}$$

where $\mathscr{W}^{-1} = \mathrm{O}(\varepsilon^2),\ \mathrm{O}(\varepsilon)$ and O(1) for large-, medium- and small-size bubbles, respectively.

5.4. Displacement condition on Γ

We now investigate the boundary condition (12). From (30) we obtain

$$\frac{\mathbf{j} \cdot \mathbf{n}}{\rho_l^\text{e} \omega u_\text{l}} = \mathrm{O}\left(\mathscr{C} \frac{\rho_\text{v}^\text{e}}{\rho_\text{l}^\text{e}} \frac{R}{u_\text{l}} \frac{p_\text{v}}{P^\text{e}} (1 - H) \right) = \mathrm{O}\left(\varepsilon^{-1} \frac{p_\text{v}}{P^\text{e}} (1 - H) \right). \tag{42}$$

In all cases $p_\text{v} = \mathrm{O}(p_\text{l}) = \mathrm{O}(\tilde{K}\theta) = \mathrm{O}(\varepsilon^2 \tilde{K})$. Therefore we obtain $p_\text{v}/P^\text{e} = \mathrm{O}(\varepsilon^2)$ for large and small-size bubbles, whereas for medium-size bubbles we have $p_\text{v}/P^\text{e} = \mathrm{O}(\varepsilon^3)$. From this result we deduce that $r/R = \mathrm{O}(\varepsilon^2)$ for very large, large or medium bubbles and $r/R = \mathrm{O}(\varepsilon)$ for small bubbles. The above estimations yield approximately similar dimensionless forms of the boundary condition (12) for large- or medium- or small-size bubbles

$$\mathrm{i}\omega \mathbf{u}_\Gamma \cdot \mathbf{n} = \mathrm{i}\omega \mathbf{u}_\text{l} \cdot \mathbf{n} - \varepsilon^2 \frac{\mathbf{j} \cdot \mathbf{n}}{\rho_l^\text{e}} = \mathrm{i}\omega \mathbf{u}_\text{v} \cdot \mathbf{n} - \varepsilon \frac{\mathbf{j} \cdot \mathbf{n}}{\rho_\text{v}^\text{e}}. \tag{43}$$

5.5. Equation of state of the vapour

Consider now the vapour state equation (7). The analysis in [2] remains valid; the relative changes of pressure, temperature and density are in all cases of the same order of magnitude. The formal scaled state equation is then

$$p_\text{v} = p_\text{v}^\text{e} \left(\frac{\rho_\text{v}}{\rho_\text{v}^\text{e}} + \frac{T_\text{v}}{T^\text{e}} \right). \tag{44}$$

5.6. Mass balances

Here also the evaluations in [2] are valid:

$$\varepsilon\rho_{\mathrm{v}} + \rho_{\mathrm{v}}^{\mathrm{e}}\,\mathrm{div}\,\mathbf{u}_{\mathrm{v}} = 0, \tag{45}$$

$$\varepsilon^2\rho_{\mathrm{l}} + \rho_{\mathrm{l}}^{\mathrm{e}}\,\mathrm{div}\,\mathbf{u}_{\mathrm{l}} = 0. \tag{46}$$

6. Macroscopic description and wave propagation

Most of the homogenization process is similar to the one conducted in [2], where phase change is neglected. To avoid repetition, only the modifications introduced by the presence of phase change are addressed here. The reader is referred to [2] to complete the analysis. The phase change effects concern the boundary conditions on Γ, only; the temperature of the bubble surface is now given by the Clausius–Clapeyron relation (10) and the thermal flux and the displacement are no longer continuous, Eqs. (11) and (12), respectively. The macroscopic momentum balances are left unchanged. In particular we note that the first-order terms in the expansions of the pressures are **y**-independent

$$p_{\mathrm{l}}^0 = p_{\mathrm{l}}^0(\mathbf{x}), \quad p_{\mathrm{v}}^0 = p_{\mathrm{v}}^0(\mathbf{x}).$$

In the cases of medium, large or very large-size bubbles we have

$$p_{\mathrm{l}}^0 = p_{\mathrm{v}}^0 = P(\mathbf{x}).$$

Each physical quantity is sought as an expansion in the form (2). The different stages of the process of homogenization are as in [2]. We first address the case of medium bubbles.

6.1. Medium-size bubbles: macroscopic description

Phase change affects the vapour temperature field only. The vapour density becomes, after recalling that $p_{\mathrm{v}}^0 = p_{\mathrm{l}}^0 = P(\mathbf{x})$

$$\frac{\rho_{\mathrm{v}}^0}{\rho_{\mathrm{v}}^{\mathrm{e}}} = \frac{P}{P^{\mathrm{e}}} - \frac{T_{\mathrm{v}}^0}{T^{\mathrm{e}}},$$

$$\rho_{\mathrm{v}}^0 = \frac{P\rho_{\mathrm{v}}^{\mathrm{e}}}{P^{\mathrm{e}}}\left[1 - g(\mathbf{y})\left(1 - \frac{1}{\gamma}\right)\right] - \frac{P}{L^{\varphi}}[1 - g(\mathbf{y})].$$

The macroscopic mass balance is obtained from the local mass balances in the vapour and the liquid (Eqs. (4A-1) and (9A-1) in [2]) and (43) at the second-order of approximation, that are respectively,

$$\mathrm{div}_y(\mathbf{u}_\mathrm{l}^1) + \mathrm{div}_x(\mathbf{u}_\mathrm{l}^0) = 0 \quad \text{in } \Omega_\mathrm{l}, \tag{47}$$

$$\rho_\mathrm{v}^0 + \rho_\mathrm{v}^\mathrm{e}[\mathrm{div}_y(\mathbf{u}_\mathrm{v}^1) + \mathrm{div}_x(\mathbf{u}_\mathrm{v}^0)] = 0 \quad \text{in } \Omega_\mathrm{v}, \tag{48}$$

$$\mathrm{i}\omega\mathbf{u}_\mathrm{l}^1 \cdot \mathbf{n} = \mathrm{i}\omega\mathbf{u}_\mathrm{v}^1 \cdot \mathbf{n} - \frac{\mathbf{j}^0 \cdot \mathbf{n}}{\rho_\mathrm{v}^\mathrm{e}} \quad \text{on } \Gamma, \tag{49}$$

where $\mathbf{j}^0 \cdot \mathbf{n}$ is given by (16)

$$L^\varphi \ \mathbf{j}^0 \cdot \mathbf{n} = -k_\mathrm{l}\, \mathbf{grad}_y\, T_\mathrm{l}^0 \cdot \mathbf{n}. \tag{50}$$

Integrating (47) and (48) on Ω_l and Ω_v, respectively, applying the divergence theorem and using (49) yields the macroscopic mass balance

$$(1-\beta)\,\mathrm{div}_x\,(\mathbf{U}_\mathrm{l}) + \beta\,\mathrm{div}_x\,(\mathbf{U}_\mathrm{v}) = \beta\frac{P}{P^\mathrm{e}}\left[-1 + G\left(1 - \frac{1}{\gamma}\right)\right] + \beta\frac{P}{L^\varphi\rho_\mathrm{v}^\mathrm{e}}[1-G] - (1-\beta)P \times \frac{\rho_\mathrm{l}^\mathrm{e} C_\mathrm{pl} T^\mathrm{e}}{\rho_\mathrm{v}^\mathrm{e} 2L^{\varphi^2}}(1-H), \tag{51}$$

where

$$\mathbf{U}_\mathrm{l} = \frac{1}{\Omega_\mathrm{l}}\int_{\Omega_\mathrm{l}} \mathbf{u}_\mathrm{l}^0\, \mathrm{d}\Omega, \quad \mathbf{U}_\mathrm{v} = \frac{1}{\Omega_\mathrm{v}}\int_{\Omega_\mathrm{v}} \mathbf{u}_\mathrm{v}^0\, \mathrm{d}\Omega, \quad G = \frac{1}{\Omega_\mathrm{v}}\int_{\Omega_\mathrm{v}} \mathbf{g}\, \mathrm{d}\Omega.$$

The last two terms in Eq. (51) are related to the presence of phase change. The macroscopic description is completed by the macroscopic momentum balances of the liquid and the vapour (see [2])

$$(1-\beta)\ \mathbf{U}_\mathrm{l} = \frac{1}{\rho_\mathrm{l}^\mathrm{e}\omega^2}\,\mathbf{grad}\,P, \tag{52}$$

$$\beta\mathbf{U}_\mathrm{v} = \frac{\beta}{\rho_\mathrm{l}^\mathrm{e}\omega^2}(\mathbf{I} + \mathbf{M}) \cdot \mathbf{grad}\,P, \tag{53}$$

where $\mathbf{M}$ is a real symmetric and ω-dependent tensor that describes the inertial coupling between liquid and bubbles. Its properties are studied in [2]. Eliminating $\mathbf{U}_\mathrm{l}$ and $\mathbf{U}_\mathrm{v}$ between 51, 52 and 53 yields, in the case of an isotropic medium for which $\mathbf{M} = M\mathbf{I}$

$$[1+\beta(1+M)]\Delta P = \rho_{\mathrm{l}}^{\mathrm{e}}\omega^2\left(\frac{\beta}{P^{\mathrm{e}}}\left[-1+G\left(1-\frac{1}{\gamma}\right)\right]+\frac{\beta}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}}[1-G]\right.$$
$$\left.+(1-\beta)\frac{\rho_{\mathrm{l}}^{\mathrm{e}}C_{\mathrm{pl}}T^{\mathrm{e}}}{\rho_{\mathrm{v}}^{\mathrm{e}}2L^{\varphi^2}}(1-H)\right)P. \tag{54}$$

As $L^{\varphi}\to\infty$, i.e. when neglecting phase change, we recover the result in [2].

6.2. Large-size bubbles: macroscopic description

The equations to be investigated for large bubbles are similar to Eqs. (47)–(50) that where obtained for medium-size bubbles. However, we have now $1-H\ll 1$ and $G\sim 1$. Therefore the mass balance (51) simplifies to

$$(1-\beta)\,\mathrm{div}_x(\mathbf{U}_{\mathrm{l}})+\beta\,\mathrm{div}_x(\mathbf{U}_{\mathrm{v}})=-\beta\frac{P}{\gamma P^{\mathrm{e}}}. \tag{55}$$

Hence, we recover the macroscopic description in [2], which is defined by (55) and

$$(1-\beta)\mathbf{U}_{\mathrm{l}}=\frac{1}{\rho_{\mathrm{l}}^{\mathrm{e}}\omega^2}\,\mathbf{grad}\,P, \tag{56}$$

$$\beta\mathbf{U}_{\mathrm{v}}=\frac{\beta}{\rho_{\mathrm{l}}^{\mathrm{e}}\omega^2}(\mathbf{I}+\mathbf{M}')\cdot\mathbf{grad}\,P, \tag{57}$$

where $\mathbf{M}'$ is a complex symmetric and ω-dependent tensor that describes both inertial and viscous coupling between liquid and bubbles (see [2]). In the case of an isotropic medium, we obtain the following macroscopic behaviour

$$[1+\beta(1+M')]\Delta P=-\beta\rho_{\mathrm{l}}^{\mathrm{e}}\omega^2\frac{P}{\gamma P^{\mathrm{e}}}. \tag{58}$$

6.3. Small-size bubbles: macroscopic description

We start from similar mass balances and boundary condition as for medium-size bubbles, Eqs. (47)–(49):

$$\mathrm{div}_y(\mathbf{u}_{\mathrm{l}}^1)+\mathrm{div}_x(\mathbf{u}_{\mathrm{l}}^0)=0\quad\text{in }\Omega_{\mathrm{l}}, \tag{59}$$

$$\rho_{\mathrm{v}}^0+\rho_{\mathrm{v}}^{\mathrm{e}}[\mathrm{div}_y(\mathbf{u}_{\mathrm{v}}^1)+\mathrm{div}_x(\mathbf{u}_{\mathrm{v}}^0)]=0\quad\text{in }\Omega_{\mathrm{v}}, \tag{60}$$

$$\mathbf{u}_{\Gamma}^1\cdot\mathbf{n}=\mathbf{u}_{\mathrm{l}}^1\cdot\mathbf{n}=\mathbf{u}_{\mathrm{v}}^1\cdot\mathbf{n}-\frac{\mathbf{j}^0\cdot\mathbf{n}}{\mathrm{i}\omega\rho_{\mathrm{v}}^{\mathrm{e}}}\quad\text{on }\Gamma. \tag{61}$$

From [2] we have now

$$p_{\mathrm{l}}^0 = p_l^0(\mathbf{x}), \quad p_{\mathrm{v}}^0 = p_{\mathrm{v}}^0(\mathbf{x}), \quad p_{\mathrm{v}}^0 - p_{\mathrm{l}}^0 = p_{\mathrm{c}}^0(\mathbf{x}),$$

$$\mathbf{U}_{\mathrm{l}} = \frac{1}{\Omega_{\mathrm{l}}} \int_{\Omega_{\mathrm{l}}} \mathbf{u}_{\mathrm{l}}^0 \, \mathrm{d}\Omega = \mathbf{U}_{\mathrm{v}} = \frac{1}{\Omega_{\mathrm{v}}} \int_{\Omega_{\mathrm{v}}} \mathbf{u}_{\mathrm{v}}^0 \, \mathrm{d}\Omega = \mathbf{U}.$$

Using (44), (27) and (41) yields

$$\frac{\rho_{\mathrm{v}}^0}{\rho_{\mathrm{v}}^{\mathrm{e}}} = \frac{p_{\mathrm{v}}^0}{P^{\mathrm{e}} + (2\sigma/R)} - \frac{T_{\mathrm{v}}^0}{T^{\mathrm{e}}} = \frac{p_{\mathrm{v}}^0}{P^{\mathrm{e}} + (2\sigma/R)} - \frac{T_\Gamma^0}{T^{\mathrm{e}}}$$

and the vaporization–condensation flux is given by

$$L^\varphi \int_\Gamma \mathbf{j}^0 \cdot \mathbf{n}_{\mathrm{l}} \, \mathrm{d}\Gamma = -\mathrm{i}\omega \, \Omega_{\mathrm{l}} \, \rho_{\mathrm{l}}^{\mathrm{e}} \, C_{\mathrm{pl}} \, T_\Gamma^0.$$

Notice that

$$\mathbf{u}_\Gamma^1 \cdot \mathbf{n}_{\mathrm{v}} = r^0 = -p_{\mathrm{c}}^0 \frac{R^2}{2\sigma},$$

$$\frac{1}{\Omega_{\mathrm{v}}} \int_\Gamma \mathbf{u}_\Gamma^1 \cdot \mathbf{n}_{\mathrm{v}} \, \mathrm{d}\Gamma = -\frac{p_{\mathrm{c}}^0}{K_{\mathrm{c}}}, \quad K_{\mathrm{c}} = \frac{2\sigma}{3R}.$$

Integrating (59) on Ω_{l}, using the divergence theorem and the first equality in (61) yields

$$\mathrm{div}_x \, \mathbf{U} = \frac{\beta}{1-\beta} \frac{p_{\mathrm{l}}^0 - p_{\mathrm{v}}^0}{K_{\mathrm{c}}}. \tag{62}$$

Integrating (60) on Ω_{v}, using the above relations for the density and the vaporization–condensation flux through Γ gives the macroscopic mass balance for the bubbles

$$\mathrm{div}_x \, \mathbf{U} = -\frac{p_{\mathrm{v}}^0}{P^{\mathrm{e}} + (2\sigma/R)} - \frac{p_{\mathrm{l}}^0 - p_{\mathrm{v}}^0}{K_{\mathrm{c}}} + \frac{p_{\mathrm{v}}^0}{L^\varphi \rho_{\mathrm{v}}^{\mathrm{e}}} - \frac{1-\beta}{\beta} \frac{\rho_{\mathrm{l}}^{\mathrm{e}} C_{\mathrm{pl}} T_\Gamma^0}{\rho_{\mathrm{v}}^{\mathrm{e}} L^\varphi} \tag{63}$$

with

$$T_\Gamma^0 = \frac{p_{\mathrm{v}}^0 T^{\mathrm{e}}}{L^\varphi \rho_{\mathrm{v}}^{\mathrm{e}}}.$$

Once again, as $L^{\varphi} \to \infty$, i.e. when neglecting phase change, we recover the result in [2]. The macroscopic description is completed by the bulk momentum balance, [2]

$$\mathbf{grad}_x p_l^0 + \frac{\beta}{1-\beta}\mathbf{grad}_x p_v^0 = \rho_l^e \omega^2 \mathbf{U}. \tag{64}$$

After eliminating p_v^0 and $\mathbf{U}$ between 62, 63 and 64, we get

$$\left(A + \frac{\beta}{1-\beta}\right)\Delta p_l^0 = \rho_l^e \omega^2 \frac{\beta}{(1-\beta)K_c}(A-1)p_l^0 \tag{65}$$

with

$$A = 1 + K_c(1-\beta)\left(\frac{1}{\rho_v^e L^{\varphi}} - \frac{1}{P^e} - \frac{1-\beta}{\beta}\frac{\mathscr{C}}{P^e}\right).$$

In the above relation, the vaporization–condensation is large. A can be approximated by

$$A \approx -K_c(1-\beta)\frac{1-\beta}{\beta}\frac{\mathscr{C}}{P^e} \gg 1.$$

By neglecting small terms, the wave equation (65) becomes

$$\frac{(1-\beta)K_c}{\beta}\Delta p_l^0 = \rho_l^e \omega^2 p_l^0.$$

That gives the following macroscopic behaviour

$$\frac{(1-\beta)K_c}{\beta}\Delta P = \rho_l^e \omega^2 P, \tag{66}$$

where P stands for the first order approximation p_l^0 of the fluid pressure perturbation.

Note that Eq. (66) was obtained from underestimated vaporization–condensation flux. Note also that the perturbation in Eq. (66) is not propagative, which is coherent with the negative rigidity obtained above. This is also in agreement with the results in [2,18].

7. Conclusion

We have used the method of double scale asymptotic expansions to derive the macroscopic modelling of acoustic waves in bubbly fluids in presence of capillarity and phase change. The macroscopic modelling strongly depends on the bubble size compared to the viscous and thermal layer thickness. In case of isotropic behaviour, acoustics in large-size bubbly fluids is described by the following wave equation:

$$[1+\beta(1+M')]\Delta P = -\beta\rho_{\mathrm{l}}^{\mathrm{e}}\omega^2\frac{P}{\gamma p^{\mathrm{e}}}. \tag{67}$$

Phase change and thermal effects are negligible. M' is a real and frequency dependent parameter which contains inertial and capillary effects.

For medium-size bubbles, phase change as well as thermal effects are present and the isotropic wave equation takes the form:

$$[1+\beta(1+M)]\Delta P = \rho_{\mathrm{l}}^{\mathrm{e}}\omega^2,$$

$$\left(\frac{\beta}{P^{\mathrm{e}}}\left[-1+G\left(1-\frac{1}{\gamma}\right)\right]+\frac{\beta}{L^{\varphi}\rho_{\mathrm{v}}^{\mathrm{e}}}[1-G]-(1-\beta)\ \frac{\rho_{\mathrm{l}}^{\mathrm{e}}C_{\mathrm{pl}}T^{\mathrm{e}}}{\rho_{\mathrm{v}}^{\mathrm{e}}2L^{\varphi^2}}(1-H)\right)P. \tag{68}$$

Finally for small-size bubbles, we obtained:

$$\frac{(1-\beta)K_{\mathrm{c}}}{\beta}\Delta P = \rho_{\mathrm{l}}^{\mathrm{e}}\omega^2 P. \tag{69}$$

As it can be seen, there is no wave propagation. We can say that the apparent rigidity of the bubbly fluid becomes negative. As the bubble radius R decreases, K_{c} increases. However, the validity of Laplace relation becomes questionable as $R < 100$ Å (see [23]). Note that the effect of the curvature on the surface tension σ is negligibly small when $R > 100$ Å.

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References

[1] J.-F. Allard, P. Herzog, D. Lafarge, M. Tamura, Recent topics concerning the acoustics of fibrous and porous materials, Appl. Acoust. 73 (3) (1993) 3–21.

[2] C. Boutin, J.-L. Auriault, Acoustics of a bubbly fluid at large bubble concentration, Eur. J. Mech. B/Fluids 12 (3) (1993) 367–399.

[3] E.L. Carstensen, L.L. Foldy, Propagation of sound through a liquid containing bubbles, J. Acoust. Soc. Am. 19 (1947) 481–501.

[4] F.E. Fox, S.R. Curley, G.S. Larson, Phase velocity and absorption measurements in water containing air bubbles, J. Acoust. Soc. Am. 27 (1955) 534–539.

[5] L. Noordzij, L. Van Wijngarden, Relaxation effects caused by relative motion on shock waves in gas–bubble liquid mixtures, J. Fluid Mech. 66 (1974) 115–143.

[6] J.C. Micaelli, Propagation d'ondes dans les écoulements à bulles à deux constituants, Doctorat d'Etat Thesis, Grenoble, 1982.

[7] R. Nigmatulin, N.S. Khabeev, Zuong Ngok Ngok Hai, Waves in liquid with vapor bubbles, J. Fluid Mech. 186 (1988) 85–117.

[8] C. Coste, C. Laroche, S. Fauve, Sound propagation in a liquid with vapour bubbles, Europhys. Lett. 11 (1990) 343–347.
[9] A. Crespo, Sound and shock waves in liquid containing bubbles, Phys. Fluids 12 (1969) 2274–2282.
[10] D.S. Drumheller, A. Bedford, A theory of liquids with vapor bubbles, JASA 67 (1980) 186–200.
[11] A. Prosperetti, Thermal effects and damping mechanisms in the forced radial oscillations of gas bubbles in liquid, J. Acoust. Soc. Am. 61 (1977) 17–27.
[12] M.J. Miksis, L. Ting, Nonlinear radial oscillations of a gas bubble including thermal effects, J. Acoust. Soc. Am. 73 (1984) 897–905.
[13] V.E. Nakoryakov, B.G. Pokusaev, I.R. Shreiber, Wave Propagation in Gas–liquid Media, CRC Press, Boca Raton, 1993.
[14] R.E. Caflisch, M.J. Miksis, G. Papanicolaou, L.J. Ting, Wave propagation in bubbly liquids at finite volume fraction, J. Fluid Mech. 160 (1985) 1–14.
[15] M.J. Miksis, L. Ting, Wave propagation in a bubbly liquid with finite-amplitude asymmetric bubble oscillations, Phys. Fluids 29 (1986) 603–618.
[16] M.J. Miksis, L. Ting, Viscous effects on wave propagation in a bubbly liquid, Phys. Fluids 30 (1987) 1683–1689.
[17] T. Levy, Propagation of waves in a mixture of fluids, Int. J. Eng. Sci. 19 (1981) 83–90.
[18] J.-L. Auriault, O. Lebaigue, Acoustic waves in a mixture of fluids with capillary effects, Int. J. Eng. Sci. 27 (1989) 1253–1265.
[19] J.-L. Auriault, C. Boutin, Acoustique d'un fluide au voisinage du point d'ébullition, CRAS, Paris, to 328, Sériellt (2000) 171–177.
[20] E. Sanchez-Palencia, Non-homogeneous Media and Vibration Theory, Lecture Notes in Physics, vol. 127, Springer, Berlin, 1980.
[21] J.-L. Auriault, Heterogeneous medium: Is an equivalent description possible?, Int. J. Engrg. Sci. 29 (1991) 785–795.
[22] J.-L. Auriault, Effective macroscopic description for heat conduction in periodic composites, Int. J. Heat Mass Transfer 26 (6) (1983) 861–869.
[23] M. Kaviany, Principles of Convection Heat Transfer, Springer, New York, 1994.